Vibrating String

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TIME APPROXIMATION – Alton: 14 hours, Mihir: 14 hours

1 Introduction

In this assignment, we explore string excitation with discrete and continuous masses. The string is fixed on one side and is attached to a vertical linear actuator moving at a set frequency on the other. We are specifically evaluating the behavior of the string mass system at its natural frequencies and predicting its mode shapes.

2 Methodology and Results

2.1 Discrete String Mass System

For the discrete string mass system we are looking at a string of length L with number of masses N being excited at one end by a linear actuator. The linear actuator oscillates according to function Uf at a frequency omega and amplitude A. For the discrete case the number of strings is relatively small(¡10).

To simulate the behavior of the string we treat the system as though it were a series of springs and masses with a small damping coefficient. The damping coefficient allows for small amounts of error to exist without causing the system to break. By framing the system as such we can find the equations of motion for the spring damper system and use them as the equations of motion for our spring mass system.

2.2 Discrete String Mass System: Modal Analysis

To find the mode shapes of our discrete system we tuned our defining equation of motion into a generalized eigenvalue problem. To do this, we simplified our equation of motion by excluding the damping we added earlier. This meant that we had a second order differential equation that was a function of just our variable U, the mass inertia coefficient M, and the spring constant K in the form:

$$MU'' + KU = B_{1\text{uf}}(t) + B_{2\text{u'f}}(t)$$

To find M we look at the relationship between our spring mass system and these coefficients. M is the mass of each of the individual masses along the string and it is independent of the masses conjunct to it. To find M we divide the total mass of the system by the number of masses giving us the mass of just an individual mass. To be able to use this coefficient in a matrix multiplication problem we multiply this mass by the identity matrix to give us a mass for each of n numbers of masses.

To find K we again look at what it represents in a spring mass system where it would represent the spring constant. For both systems k is our resistance to displacement. In the case of the string, the resistance to displacement comes from the tension in the string in the adjacent sections of string. This is why the tension is divided by the distance between the masses and then multiplied by the Laplacian matrix to encompass the diminishing effect the conjunct masses have.

Once we have the values for M and K we now have a generalized eigenvalue problem in which solving using Matlab's eig function yields us the resonant frequency and y positions of the masses for each mode shape.

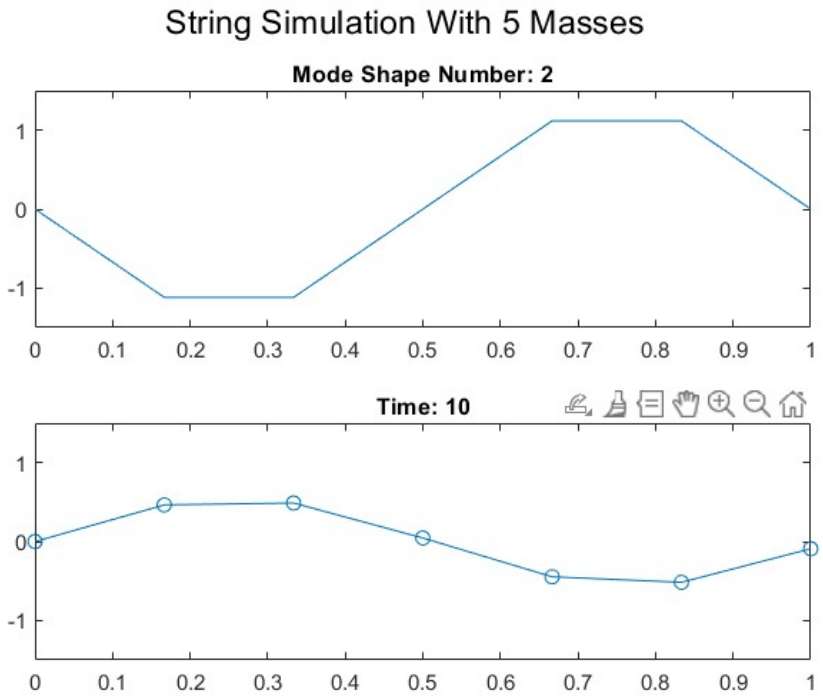

Figure 1: A subplot containing our prediction for the mode shape(above) and our string mass simulation(below). A potentially more helpful visualization is this video `https://youtu.be/kZ0LHBj-Wsk`. In both these instances it can be seen that our predicted mode shape matches the behavior of our string simulation.

2.3 Continuous String Mass System

To simulate a continuous system we made the number of masses spread across the string much greater than the length. While this is still technically discrete it is a close approximation of a continuous system, as we verified in our results section.

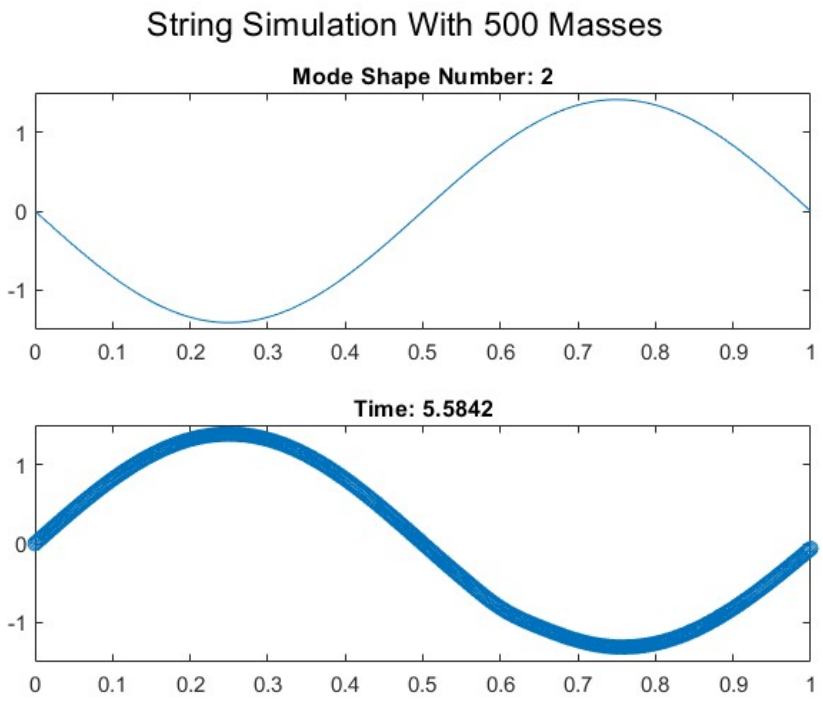

Figure 2: A subplot containing our prediction for the mode shape(above) and our string mass simulation(below). This figure is the same as the previous discrete example except this time the number of masses is much greater creating an approximation of a continuous system.

2.4 Continuous String Mass System with Traveling Waves

For our string mass system with a traveling wave, we used the same continuous approximation that is described above but changed the excitement from the linear actuator to be just a single pulse. The speed of the single pulse determined the shape of the traveling wave. To track the peak of the standing wave as it moved up and down our string system we used the wave speed coefficient. We found the wave speed in units per second using the wave speed equation:

$$c = \sqrt{\frac{T_{\mathrm{f}}}{p}}.$$

and then multiplying the wave speed coefficient c by the time step to find the distance the crest of the wave would move for each time step. By then moving by this amount in the direction the wave was traveling we were able to track the crest.

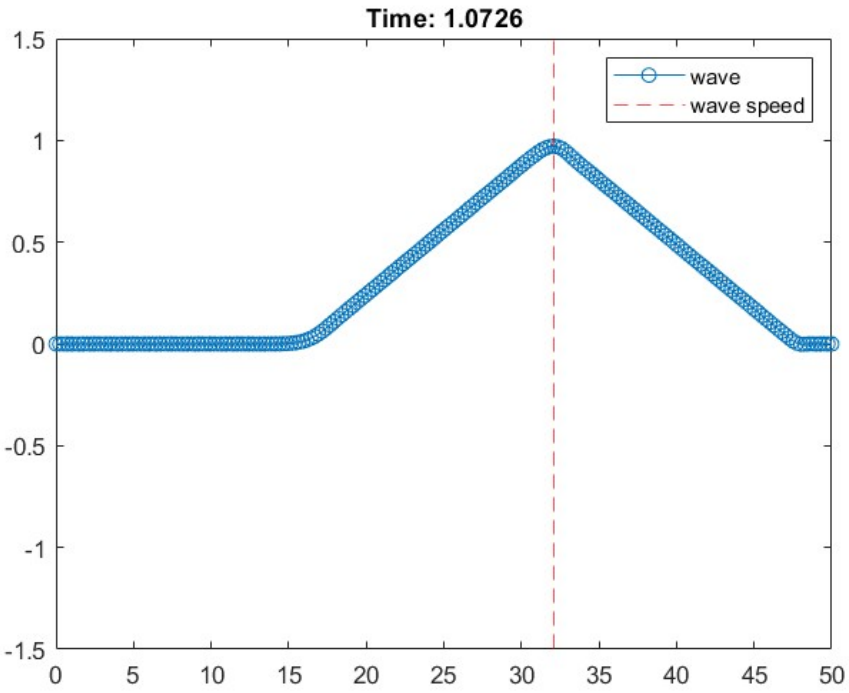

Figure 3: A plot containing a traveling wave and a vertical line moving at the wave speed constant. The vertical wave speed line intersects the traveling wave at its peak and continuous to do so as the wave travels up and down the string. This is more easy represented by this video: `https://youtu.be/kZOLHBj-Wsk`. This confirms that the wave speed constant accurately describes the speed of the wave.

2.5 Continuous String Mass System: Modal Analysis

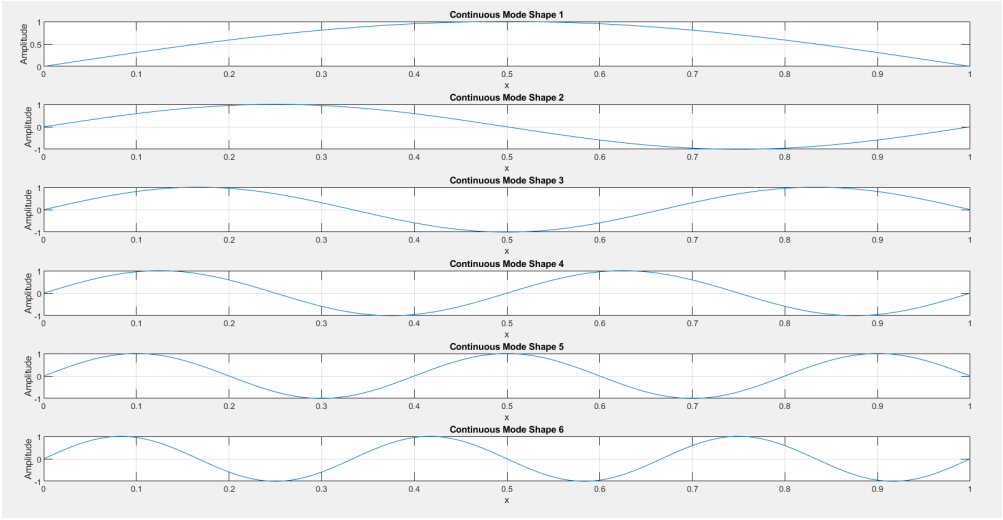

Figure 4: A subplot containing all the different mode shapes

This figure has a series of distinct modes that our vibrating string might naturally adopt. Each plot shows how the system's shape changes along it's length. For example, mode shape 1 is the simplest pattern and

represents the lowest frequency and most fundamental mode. The modes progressively get more complex because more internal nodes and higher natural frequencies become present.

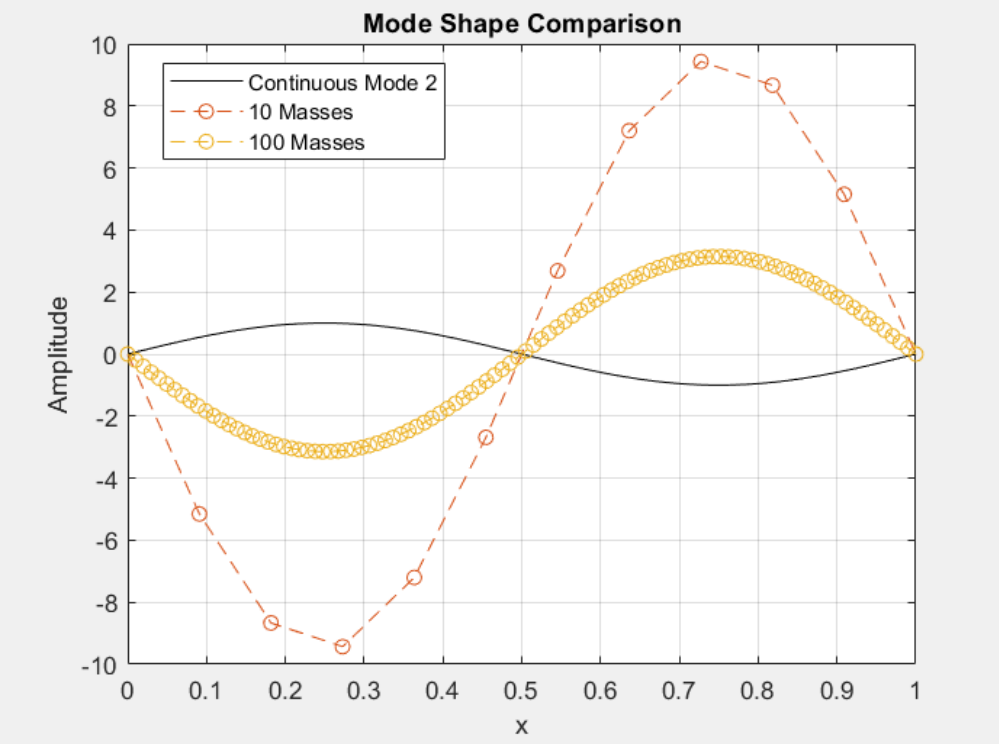

Figure 5: Compare the continuous function to plots representing different number of masses

For our mode shape comparison plot, we chose to use Mode 2. This plot compares our theoretical continuous mode shape of the vibrating system to the numerical approximations we made modeling the same system as a chain of discrete masses. With fewer masses (10), the mode shape was more angular and deviated a lot from the smooth continuous curve, so it was a coarse approximation. Increasing the number of masses (100) resulted in a better fit and a curve that more closely approached the continuous solution's shape.

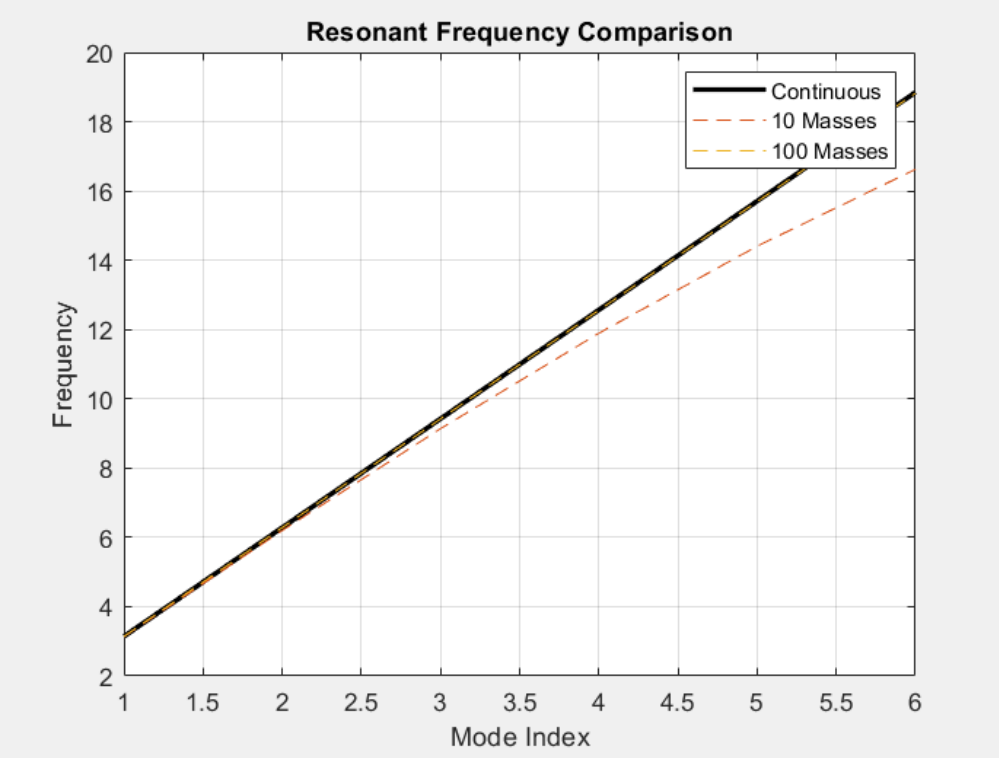

Figure 6: resonant frequency comparison with the different number of masses

This plot shows how the natural frequencies of a vibrating system are different when they are modeled as an ideal continuous system versus the approximation with a finite number of masses. Each point on the horizontal axis represents a different mode of vibration, while the vertical axis shows the corresponding frequency for that mode.

Like in the other plot, the solid line is the theoretical continuous, but here it represents frequency. The dashed lines, represent the discretized models (with 10 or 100 masses), start out similar to the continuous curve but begin to deviate for higher modes. The model with only 10 masses is less accurate and shows more of a difference, while the model with 100 masses aligns more closely with the continuous case, indicating a better approximation of the true frequencies.

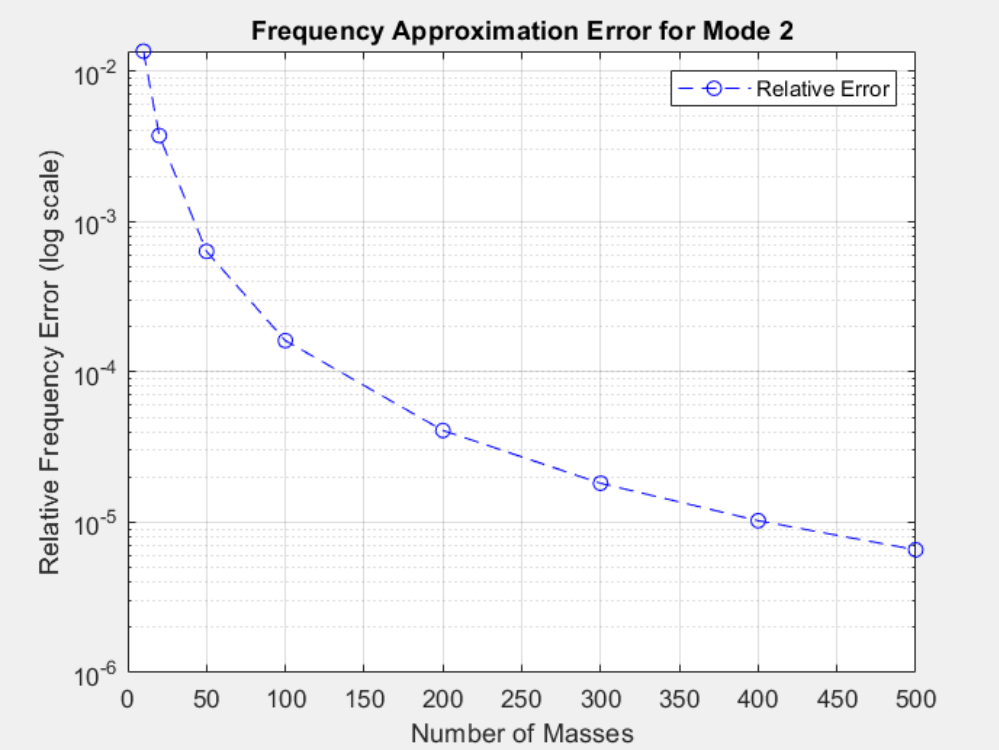

Figure 7: Frequency error approximation

As you move from left to right, the number of masses grows, and the curve steadily drops downward, indicating that the relative error is decreasing. This trend shows that starting with a small number of masses leads to a large deviation from the correct frequency. However, as more masses are added, the discretized system more closely emulates the continuous system's behavior, and the frequency prediction becomes more accurate.

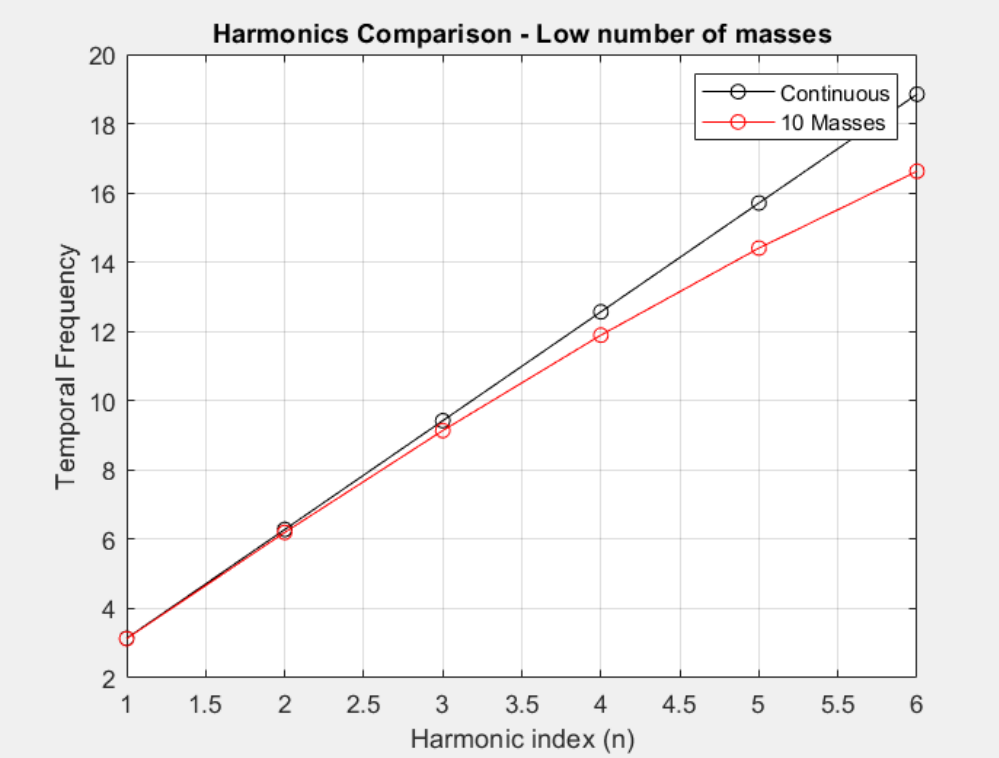

Figure 8: Harmonic frequency with a low number of masses

This figure compares the harmonic frequencies obtained from the continuous model to those predicted by a discrete model using a small number of mamasses (10). (1e horizontal axis lists the mode number, and the vertical axis shows the corresponding temporal frequency.

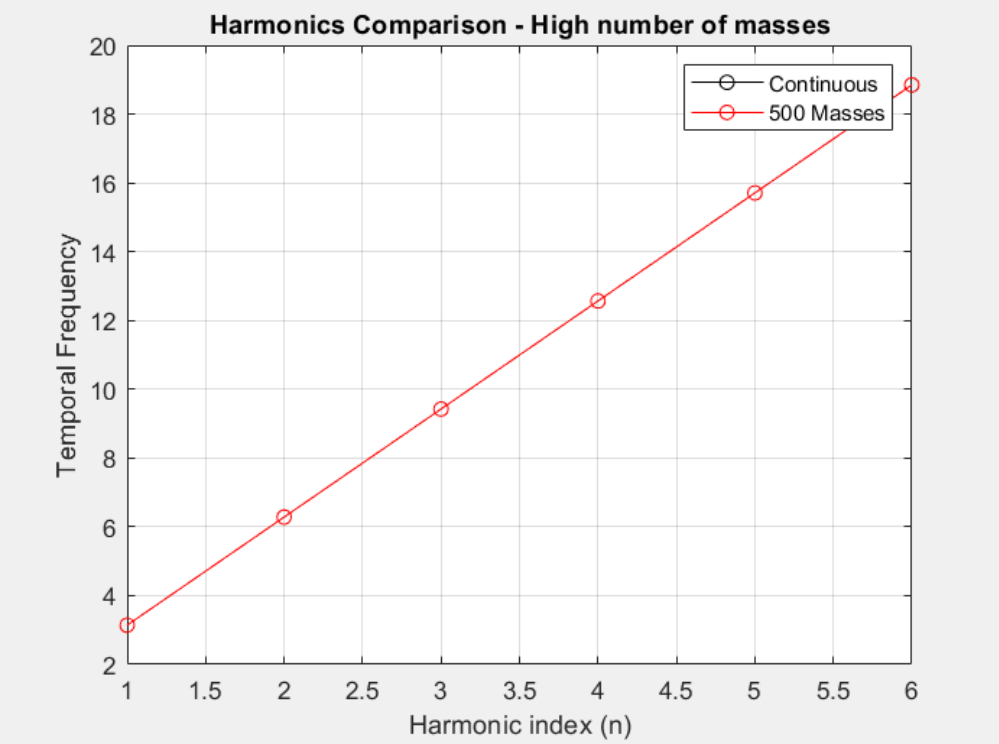

Figure 9: Harmonic frequency with a high number of masses

This figure is the same harmonic comparison but with a higher number of masses. The larger number of masses very closely matches the continuous function, which make sense in the context of our model.

3 Link to Our Code

Here is the link to our code and videos of our animations: `https://drive.google.com/drive/folders/1ZSur-rR4ci0e-YzheueLIlSR-niFSSDD?usp=sharing`